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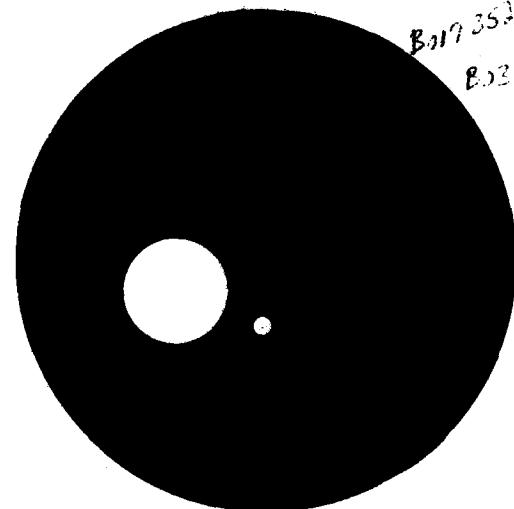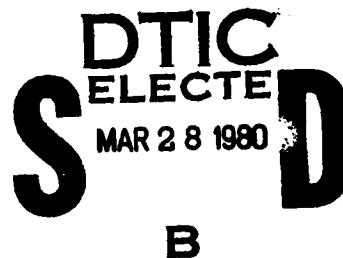
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ANOTHER LOOK AT ITERATIVE METHODS
FOR ELLIPTIC DIFFERENCE EQUATIONS

by

Seymour V. Parter

and

Michael Steuerwalt

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~~and~~

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ANOTHER LOOK AT ITERATIVE METHODS

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ABSTRACT

Iterative methods for solving elliptic difference equations have
received new attention because of the recent advent of novel computer
architectures and a growing interest in three-dimensional problems.
The fundamental characteristic of an iterative method is its rate of
convergence. We present here, in the context of the model problem in
two and three dimensions, a very simple theory for determining the
rates of convergence of block iterative schemes. This theory is easily
extended to general domains, general elliptic problems, and higher
dimensions.

Introduction

Some 15-20 years ago there was considerable interest in iterative methods for elliptic difference equations - see [13],[14],[18],[19],[20], [21],[10]. More recently there has been a greater emphasis on direct methods for these sparse matrices, as shown by [6],[7],[15],[16].

However, the advent of new computer architectures — "vector machines" and "parallel processors" — and a renewed interest in three-dimensional problems make it desirable to reconsider the analysis of iterative methods, as in [3],[8],[9],[17].

A fairly general theory for estimating the rates of convergence of iterative methods for elliptic difference equations was developed in [13]. However, partly because of the generality of that work (variable coefficients, general domains, etc.), it is by no means a transparent discussion. Because of this, and the interest in a particular block iterative scheme ($k \times k$ blocks), D. Boley, B. Buzbee and S.V. Parter [3] gave a relatively direct discussion of the basic ideas for the case of the model problem - the Poisson equation in a square. Their presentation was based on the relatively strong estimates of J. Nitsche and J.C.C. Nitsche [12] and A. Brandt [4].

In this report we give another variant of this general approach to the problem of estimating the rates of convergence. Our new presentation avoids the estimates of [4] and [12]. For the model problem in two or three dimensions this is a small thing. However, the estimates of [12] have never been extended to general regions and cannot hold in more than three dimensions. On the other hand, the approach taken here can easily

be extended to general domains, general elliptic equations, and any dimension. Thus, in addition to solving some particular problems of interest we also give a development which could be used in the general case.

In section 2 we describe the model problem (in two and three dimensions). In section 3 we develop the general theory for these model problems. In section 4 we describe some particular iterative schemes. These are:

i) <u>In two dimensions</u> - k-line blocks and k×k square blocks.

ii) <u>In three dimensions</u> - k-plane blocks, k×k-line blocks and k×k×k cubic blocks.

Finally, in section 5 we obtain the spectral radius of the block Jacobi scheme associated with each of these block structures. The spectral radii of the associated Gauss-Seidel and SOR schemes are then determined in the usual way (these schemes all satisfy block property A- see [1],[21]) from the formulae (see [1],[20],[18])

1.1) $$\rho_{GS} = \rho_J^2$$

1.2) $$(\rho_\omega + \omega - 1)^2 = \omega \rho_\omega \rho_J^2$$

where ρ_{GS}, ρ_ω and ρ_J denote the spectral radii of the Gauss-Seidel, SOR and Jacobi iterative schemes respectively.

We are indebted to Bill Buzbee for his continued encouragement in this work.

2. The Model Problems

Let

2.1a) $\Omega(2) \equiv \{(x,y); \ 0<x,y<1\}$,

2.1b) $\Omega(3) \equiv \{(x,y,z); \ 0<x,y,z<1\}$.

Let P be a fixed integer and set

$$h = \frac{1}{P+1} .$$

Consider the sets of interior mesh points

2.2a) $\Omega(2,h) \equiv \{(x_i,y_j) = (ih,jh)\}$, $1 \leq i, j \leq P$,

2.2b) $\Omega(3,h) \equiv \{(x_i,y_j,z_n) = (ih,jh,nh)\}$, $1 \leq i, j, n \leq P$,

as well as the boundary mesh points

2.3a) $\partial\Omega(2,h) \equiv \{(x_i,y_j) = (ih,jh; \ i \ \text{or} \ j = 0 \ \text{or} \ P+1\}$,

2.3b) $\partial\Omega(3,h) \equiv \{(ih,jh,nh); \ i \ \text{or} \ j \ \text{or} \ n = 0 \ \text{or} \ P+1\}$.

As usual, we define the discrete Laplace operators by

   Two-dimensional Case: for $1 < i, j < P$

2.4a) $(\Delta_h(2)U)_{ij} = \dfrac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \dfrac{U_{i,j+1} - 2U_{i,j} + U_{i,j+1}}{h^2}$,

Three-dimensional Case: for $1 < i, j < P$

2.4b)
$$(\Delta_h(3)U)_{i,j,n} = \frac{U_{i+1,j,n} - 2U_{i,1,n} + U_{i-1,j,n}}{h^2}$$

$$+ \frac{U_{i,j+1,n} - 2U_{i,j,n} + U_{i,j-1,n}}{h^2} + \frac{U_{i,j,n+1} - 2U_{i,j,n} + U_{i,j,n-1}}{h^2}$$

Note: While U is to be defined on the entire mesh region $\Omega(2,h) \cup \partial\Omega(2,h)$ or $\Omega(3,h) \cup \partial\Omega(3,h)$, $\Delta_h(2)$ and $\Delta_h(3)$ are defined only on the interior mesh points, $\Omega(2,h)$ or $\Omega(3,h)$. Also we define the usual difference operators

2.5a)
$$[\nabla_x U]_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{h} , \; 1 \le j \le P, \; 0 \le i \le P$$

2.5b)
$$[\nabla_y U]_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{h} , \; 0 \le j \le P, \; 1 \le i \le P.$$

In fact, we will use the symbols ∇_x, ∇_y, ∇_z to designate the forward divided difference quotients in the x, y and z directions, in all dimensions. This usage should cause no confusion.

The basic problem is: Given grid vectors F and G, and $m = 2$ or 3, find a grid vector U such that

2.6a)
$$\Delta_h(m)U = F \quad \text{in} \; \Omega(m,h).$$

2.6b)
$$U = G \quad \text{on} \; \partial\Omega(m,h).$$

After an ordering of the points (x_i, y_j) or (x_i, y_j, z_n) is determined we let A be the matrix representation of $-h^2 \Delta_h(m)$; symbolically, we write

2.7) $\qquad A \sim -h^2 \Delta_h(m).$

As we have already noted, $\Delta_h(m)$ maps vectors with $P^m + 2mP^{m-1}$ components into vectors with P^m components. The matrix A is actually a P^m by P^m matrix. The known boundary values, G, are put on the right-hand-side. In this way the difference equations (2.6a), (2.6b) take the form

2.8) $\qquad AV = \widetilde{F}$

where the $\sim$ over F is meant to indicate both the result of ordering the components of $-h^2 F$ and the necessary modifications of F required by the G terms. In any case, every vector V with P^m components may be thought of as a grid vector which also satisfies

2.9) $\qquad V = 0$ on $\partial\Omega(m,h).$

An iterative method for the solution of 2.8) is determined by a "splitting"

2.10a) $\qquad A = M - N.$

Equation (2.8) is then

2.10b) $\qquad MV = NV + \widetilde{F}.$

After choosing a first guess V^0, one obtains $V^1, V^2, \ldots, V^k, \ldots$ from

2.11) $\qquad MV^{k+1} = NV^k + \widetilde{F}$.

Let

2.12) $\qquad \rho = \max\{|\lambda|; \det(\lambda M - N) = 0\}$.

It is well known that the iterates V^k converge to the unique solution V of (2.8), independently of V^0, if and only if

2.13) $\qquad \rho < 1$.

The first problem studied in this report is: Find the asymptotic behaviour of ρ as $h \to 0$.

Remark: Of course, for every λ which is a generalized eigenvalue (i.e., $\det(\lambda M - N) = 0$) there is a vector $U \neq 0$ such that

2.14) $\qquad \lambda MU = NU$.

3. A General Approach

We make some assumptions about the splitting (2.10a).

A.1) $M = M^*$ and is positive definite.

A.2) $\rho = \max\limits_{x \neq 0} \dfrac{\langle Nx,x \rangle}{\langle Mx,x \rangle}$

where

$$\langle x,y \rangle = x^t \bar{y} = \sum x_{ij}\bar{y}_{ij} \quad (\text{or} = \sum x_{ijn}\bar{y}_{ijn}) .$$

Note: $N = N^*$ because $A = A^*$ and $M = M^*$; as is well known [5], the generalized eigenvalues are all real and

$$\rho = \max\limits_{x \neq 0} \frac{|\langle Nx,x \rangle|}{\langle Mx,x \rangle} .$$

The force of the assumption (A.2) therefore is that $\max|\lambda|$ occurs for a positive eigenvalue $\lambda = \rho$.

A.3) There is a positive constant N_0, independent of h, such that

$$\|N\|_\infty \leq N_0 .$$

Here

$$\|N\|_\infty = \sup\{|(NU)_{ij}|; \ |u_{ij}| \leq 1\} .$$

Finally we come to the main new concept.

A.4) There are positive constants q, D, independent of h, such that   if U is a grid vector satisfying

$$U = 0 \quad \text{on} \quad \partial\Omega(m,h)$$

then

3.1) $\qquad \langle NU,U \rangle = q \langle U,U \rangle + E \, ,$

where

3.2) $\qquad |E| \leq Dh[\langle U,U \rangle - \langle \Delta_h U,U \rangle] \, .$

Remark: As one might imagine, the determination of q and the verification of (A.4) are the important technical aspects of this analysis when applied to any particular case. However, as we shall see, it is not too difficult.

Lemma 3.1: Suppose the splitting (2.10a) satisfies (A.1) and (A.2). Then the method is convergent, that is,

3.2) $\qquad \rho < 1 \, .$

Proof: Let U be the eigenvector associated with ρ . Then $\langle NU,U \rangle \geq 0$. Since $M = A + N$ and A is positive definite, we have

$$0 \leq \rho = \frac{\langle NU,U \rangle}{\langle MU,U \rangle} = \frac{\langle NU,U \rangle}{\langle AU,U \rangle + \langle NU,U \rangle} < 1 \, .$$

The basic result of this section is

Theorem 3.1: Suppose the splitting (2.10a) satisfies the conditions (A.1), (A.2), (A.3) and (A.4). Then

3.3) $\qquad \rho = 1 - \frac{m\pi^2 h^2}{q} + O(h^3) \, .$

Proof: Let U be the grid vector

3.4a) $\qquad U_{ij} = (\sin i\pi h)(\sin j\pi h), \quad m = 2$,

3.4b) $\qquad U_{ijn} = (\sin i\pi h)(\sin j\pi h)(\sin n\pi h), \quad m = 3$.

Then $U = 0$ on $\partial\Omega$. The following facts are well-known (see [18], particularly page 202).

3.5) $\qquad h^m \langle U, U \rangle = \left[\frac{1}{2}\right]^m \left[\frac{P}{P+1}\right]^m, \quad m = 2, 3$.

3.6) $\qquad \dfrac{\langle AU, U \rangle}{\langle U, U \rangle} = 2m(1 - \cos \pi h) = m\pi^2 h^2 [1 - \frac{1}{12}(\pi h)^2 + 0(h^4)]$.

Furthermore, for all grid vectors $V \neq 0$ which vanish on $\partial\Omega(m,h)$ we have the inequality

3.7) $\qquad \dfrac{-\langle \Delta_h V, V \rangle}{\langle V, V \rangle} = \dfrac{1}{h^2} \dfrac{\langle AV, V \rangle}{\langle V, V \rangle} \geq \dfrac{2m(1 - \cos \pi h)}{h^2} = m\pi^2 + 0(h^2)$.

Because $M = A + N$,

$$\rho \geq \frac{\langle NU, U \rangle}{\langle MU, U \rangle} = \frac{\langle NU, U \rangle}{\langle AU, U \rangle + \langle NU, U \rangle} \ .$$

Applying (A.4) and (3.6) we have

$$\langle NU, U \rangle = q\langle U, U \rangle + E$$

where

$$|E| \leq Dh[1 + m\pi^2 + 0(h^2)]\langle U, U \rangle \ .$$

Thus

$$\rho \geq \frac{1}{1 + \frac{\langle AU,U \rangle}{q \langle U,U \rangle + E}} \ .$$

Using (3.6) we obtain

3.8) $\qquad \rho \geq 1 - \frac{m\pi^2 h^2}{q} + 0(h^3) \ .$

We now turn to the proof of the reverse inequality.

Let U be the eigenvector associated with ρ and normalized
so that

3.9) $\qquad h^m \langle U,U \rangle = 1 \ .$

Then

$$\rho MU = NU$$

$$\rho AU = \rho(M-N)U = (1-\rho)NU \ .$$

Hence

3.10a) $\qquad -\Delta_h(m)U = \mu NU$

where

3.10b) $\qquad \mu = (1-\rho)/\rho h^2$

From lemma 3.1 and (3.8) we see that

3.11a) $\qquad 0 < \mu \leq \frac{m\pi^2}{q} + 0(h) \ ,$

and

3.11b)
$$\limsup_{h \to 0} \mu \le \frac{m\pi^2}{q} \; .$$

Moreover, the theorem will be proven if we show that

3.12)
$$\mu \ge \frac{m\pi^2}{q} + O(h) \; .$$

From (3.10a), (A.3), and (3.11b) we see that when h is sufficiently small

3.13)
$$-h^m \langle \Delta_h(m)U, U \rangle \le [\frac{2m\pi^2}{q} N_0] h^m \langle U, U \rangle \; .$$

But (3.10a) and (A.4) show that

3.14a)
$$h^m \langle -\Delta_h(m)U, U \rangle = \mu \, [q h^m \langle U, U \rangle + E],$$

where

3.14b)
$$|E| \le hD[h^m \langle U, U \rangle - h^m \langle \Delta_h(m)U, U \rangle] \; .$$

Substituting (3.13) into (3.14b) gives

3.14c)
$$|E| \le hD[1 + \frac{2m\pi^2}{q} N_0][h^m \langle U, U \rangle] \; .$$

Using (3.7), (3.9) and (3.14c) in (3.14a) we have

$$\frac{-h^m \langle \Delta_h(m)U, U \rangle}{h^m \langle U, U \rangle} = \mu[q + \widetilde{E}] \ge m\pi^2 + O(h)$$

where

$$|\widetilde{E}| \leq hD[1 + \frac{2m\pi^2}{q} N_0] .$$

Thus

$$\mu \geq \frac{m\pi^2}{q} + O(h)$$

and the theorem is proven.

4. Some Iterative Methods

In this section we describe the basic block structure associated with the iteratives schemes of interest.

Consider a system of linear equations

4.1) $Ax = y$

where A is an $R \times R$ matrix. A block Jacobi iterative method for the solution of (4.1) is completely described by describing the partition of the R-vector x into blocks. Specifically, suppose that we imagine all R-vectors U partitioned into block vectors of the form

4.2) $U = (U_1, U_2, \ldots U_R,)^t$

where each U_j is itself an R_j-vector. Corresponding to this partition of vectors the matrix A is naturally partitioned in blocks

$$A = (A_{ij})$$

where each A_{ij} is itself a matrix. In particular, each diagonal block A_{ii} is a square $R_i \times R_i$ matrix. The block Jacobi iterative scheme associated with this block structure is now given by

4.3) $A_{jj} X_j^{(\nu+1)} = - \sum_{j \neq s} A_{js} X_s^{(\nu)} + Y_j$.

In terms of the discussion in sections 2 and 3, we have a splitting (2.10a) with

4.4) $M = \text{diagonal } (A_{jj})$.

The iterative schemes discussed in this report are described by the following block decompositions. Let k be a fixed integer. We consider the case where k divides P, i.e. there is an integer $Q \geq 2$ such that

4.5) $\qquad P = kQ$.

Of course, $Q \rightarrow \infty$ as $P \rightarrow \infty$ $(h \rightarrow 0)$ and vice versa.

I.) <u>Two-dimensional Problems</u>:

    1.) <u>k-line iterative scheme</u>. This scheme is described in detail in [13],[14]. Each "block" consists of the unknowns U_{ij} associated with the points on k consecutive horizontal (or vertical) lines. For example, the s'th block consists of the values

4.6) $\qquad \{U_{ij}; \ 1 \leq i \leq P, \ k(s-1)+1 \leq j \leq ks\}$.

    2.) <u>k×k block iterative scheme</u>. This scheme is described in [3]. Each block consists of the unknowns U_{ij} associated with the points in a k×k square. It is easiest to describe this block with a double index (r,s). The (r,s) block consists of the values

4.7) $\qquad \{U_{ij}; \ k(r-1)+1 \leq i \leq kr, \ k(s-1)+1 \leq j \leq ks\}$.

II.) <u>Three-dimensional Problems</u>:

    1.) <u>k-plane iterative scheme</u>. In this scheme each block consists of the unknowns U_{ijn} associated with the points on k consecutive planes. In this case the blocks are associated with a single index, say s.

The s'th block consists of the values

4.8) $\{U_{ijn};\ 1 \leq i,\ j \leq P,\ k(s-1)+1 \leq n \leq ks\}$.

2.) <u>k×k-line iterative scheme</u>. In this scheme each block consists of the unknowns U_{ijn} associated with the points on a k×k square of lines in the (say) z-direction. In this case the blocks carry a double index, say (r,s). The (r,s) block consists of the values

4.9) $\{U_{ijn};\ 1 \leq n \leq P,\ k(r-1)+1 \leq i \leq kr,\ k(s-1)+1 \leq j \leq ks\}$.

3.) <u>k×k×k block iterative scheme</u>. In this scheme each block consists of the unknowns U_{ijn} associated with a k×k×k (cubic) block of points. In this case each block is associated with a triple index, say (r,s,t). The (r,s,t) block consists of the values

4.10) $\{U_{ijn};\ k(r-1)+1 \leq i \leq kr,\ k(s-1)+1 \leq j \leq ks,\ k(t-1)+1 \leq n \leq kt\}$.

It is tedious but not difficult to write the representation of A, M and N for each of the Jacobi block iterative schemes associated with these choices of the block. We shall give a unified development of these schemes. However, for the moment, for illustrative purposes, we sketch the k-line scheme I.1.

Let L_k be the k×k tridiagonal matrix

4.11a) $L_k = [-1,4,-1]_k$.

Let T be the kP × kP block tridiagonal matrix

4.11b) $T = [-I_k, L_k, -I_k]_P$,

where I_k is the k×k identity matrix.

Let R be the $kP \times kP$ matrix

4.11c) $\qquad R = \begin{bmatrix} & I_k \\ \bigcirc & \end{bmatrix}$.

Then

4.12a) $\qquad A = M-N$

where

4.12b) $\qquad M = \text{diagonal } (T)$,

and N is the block tridiagonal matrix

4.12c) $\qquad N = [R,0,R^t]$.

A unified approach is provided by considering the 1-dimensional operator $\widetilde{N}$ acting on vectors

4.13) $\qquad V = \{V_1, V_2, \ldots V_p\}^t$

as follows for $1 \leq s \leq Q-1$, $0 \leq j \leq k-1$:

4.14) $\qquad (\widetilde{N}V)_{ks+j} = \begin{cases} 0, & 2 \leq j \leq k-1 , \\ V_{ks+1}, & j = 0 , \\ V_{ks}, & j = 1 . \end{cases}$

Let N_x, N_y, N_z be the operators which act on grid vectors in 2 or 3 dimensions in the following manner: N_x acts on U only in the x direction, and in that direction acts as $\widetilde{N}$. Similarly N_y and N_z

act on U in the y or z directions respectively. For example: in three dimensions, for $1 \leq i, j \leq P$ and $0 \leq n \leq k-1$,

4.15) $$(N_z U)_{i,j,kt+n} = \begin{cases} 0, & 2 \leq n \leq k-1 \\ U_{i,j,kt+1}, & n = 0 \\ U_{i,j,kt}, & n = 1 \end{cases}$$

With these operators it is relatively easy to describe the matrix N of the five splittings described above.

Theorem 4.1: Let $k \geq 2$. Then

For the two-dimensional problem, k-line scheme:

4.16a) $$N = N_y$$

For the two-dimensional problem, k×k block scheme:

4.16b) $$N = N_x + N_y$$

For the three-dimensional problem, k-plane scheme:

4.17a) $$N = N_z$$

For the three-dimensional problem, k×k-line scheme:

4.17b) $$N = N_x + N_y$$

For the three-dimensional problem k×k×k (cubic) block scheme:

4.17c) $$N = N_x + N_y + N_z$$

Proof: The proofs of these formulae are given directly by computation and inspection.

Theorem 4.2: Let $k = 1$; then the dominant eigenvalue of the Jacobi iterative scheme associated with each of the above block schemes is given directly by the theory of tensor products. They are:

The two-dimensional problem, k-line scheme:

4.18a) $\qquad \rho(1-L) = 1 - \pi^2 h^2 + 0(h^4)$.

The two-dimensional problem, 1×1 block, i.e. "point" scheme:

4.18b) $\qquad \rho(1-B) = \cos h = 1 - \frac{1}{2} \pi^2 h^2 + 0(h^4)$

The three-dimensional problem, 1-plane scheme:

4.19a) $\qquad \rho(1-P) = \frac{2 \cos\pi h}{6 - 4 \cos\pi h} = 1 - \frac{3}{2} \pi^2 h^2 + 0(h^4)$

The three-dimensional problem, 1×1-line scheme:

4.19b) $\qquad \rho(1^2-L) = \frac{4 \cos\pi h}{6 - 2 \cos\pi h} = 1 - \frac{3}{4} \pi^2 h^2 + 0(h^4)$

The three-dimensional problem, $1 \times 1 \times 1$ block, i.e. "point" scheme:

4.19c) $\qquad \rho(1^3-B) = \cos\pi h = 1 - \frac{1}{2} \pi^2 h^2 + 0(h^4)$

Proof: These formulae follow immediately from the standard tensor product formulation - see [10].

5. Rates of Convergence: $k \geq 2$

In this section we consider the model problem (2.6a), (2.6b).
We study the five block Jacobi schemes described in section 4
(with $k \geq 2$). We determine the appropriate constants q and D so
that we satisfy (A.4) and thus may apply Theorem 3.1 to obtain the
dominant eigenvalues of these iterative schemes.

Lemma 5.1: Let $0 < hk < 1$ and let

$$5.1) \qquad V = (V_1, V_2, \ldots V_p)^t .$$

Let $\widetilde{N}$ be given by (4.14). Then

$$5.2a) \qquad \langle V, \widetilde{N}V \rangle = \frac{2}{k} \langle V, V \rangle + 2\varepsilon$$

where

$$5.2b) \qquad |\varepsilon| \leq h \langle V, V \rangle + 2h \langle \nabla_x V, \nabla_x V \rangle .$$

Proof: For every j, $1 \leq j \leq k$ we have

$$5.3) \qquad V_{ks+1} = V_{ks+j} - h \sum_{\ell=1}^{j-1} (\nabla_x V)_{ks+\ell} ,$$

$$5.4) \qquad V_{ks} = V_{ks+j} - h \sum_{\ell=0}^{j-1} (\nabla_x V)_{ks+\ell} .$$

Of course, the sum in (5.3) is empty if $j = 1$.

In any case

5.5) $$V_{ks+1}V_{ks} = (V_{ks+j})^2 - hV_{ks+j}(E_1+E_2) + h^2E_1E_2$$

where

$$|E_\alpha| \leq \sum_{\ell=0}^{k-1} |(\nabla_x V)_{ks+\ell}| \ , \qquad \alpha = 1,2 \ .$$

Summing (5.5) on j and dividing by k, we have

5.6) $$V_{ks+1}V_{ks} = \frac{1}{k}\sum_{j=1}^{k}(V_{ks+j})^2 + \frac{1}{k}\widetilde{E}_s$$

where

$$|\widetilde{E}_s| \leq 2h(\sum_{j=1}^{k}|V_{ks+j}|) \sum_{\ell=0}^{k-1}|(\nabla_x V)_{ks+\ell}|$$

$$+ kh^2 (\sum_{\ell=0}^{k-1}|(\nabla_x V)_{ks+\ell}|)^2 \ .$$

Thus we can estimate $\widetilde{E}_s$ by

5.7) $$|\widetilde{E}_s| \leq hk\left[\sum_{j=1}^{k}(V_{ks+j})^2 + (1+hk)\sum_{\ell=0}^{k-1}(\nabla_x V)_{ks+\ell}^2\right]$$

Now (5.2) follows from the form of $\widetilde{N}$ and summation on

$s = 0,1,\ldots Q-1$.

We recall two basic identities.

<u>Lemma 5.2</u>: Let $m = 2$. Then

5.8a) $\qquad -\langle U, \Delta_h U \rangle = \langle \nabla_x U, \nabla_x U \rangle + \langle \nabla_y U, \nabla_y U \rangle$.

Let $m = 3$. Then

5.8b) $\qquad -\langle U, \Delta_h U \rangle = \langle \nabla_x U, \nabla_x U \rangle + \langle \nabla_y U, \nabla_y U \rangle + \langle \nabla_z U, \nabla_z U \rangle$.

<u>Proof</u>: These results follow directly from summation by parts (see [1],[18]).

<u>Theorem 5.1</u>: Consider the two-dimensional block Jacobi iterative schemes (I.1), (I.2) described by the blocks (4.6) and (4.7) respectively. Let $\rho(kL)$ and $\rho(kB)$ denote their respective dominant eigenvalues. Then

5.9a) $\qquad \rho(kL) = 1 - k\pi^2 h^2 + O(h^3)$,

5.9b) $\qquad \rho(kB) = 1 - \frac{1}{2} k\pi^2 h^2 + O(h^3)$.

<u>Proof</u>: Because of Theorem 4.2 we need only consider the case $k \geq 2$. From the block structure we see that $M = M^*$ and is positive definite. Moreover, block property A holds and thus (A.2) holds. It is obvious from Theorem 4.1 that

$$\| N \|_\infty \leq 3 .$$

Thus, it is only necessary to determine q and D .

For the k-line scheme we apply Theorem 4.1, specifically (4.16a). Applying lemma 5.1 we have

$$\langle U,NU \rangle = \frac{2}{k} \langle U,U \rangle + E$$

where

$$|E| \leq 2h[\langle V,V \rangle + \langle \nabla_y V, \nabla_y V \rangle] \ .$$

Using (5.8a) we have

$$q = \frac{2}{k} \ , \quad D = 2 \ , \quad m = 2$$

and (5.9a) follows at once.

For the $k \times k$ block scheme we obtain

$$q = \frac{4}{k} \ , \quad D = 2 \ , \quad m = 2$$

and (5.9b) follows.

<u>Theorem 5.2</u>: Consider the three-dimensional block Jacobi schemes (II.1), (II.2) and (II.3) described by the blocks (4.8), (4.9) and (4.10) respectively. Let $\rho(kP)$, $\rho(k^2L)$ and $\rho(k^3B)$ denote their respective dominant eigenvalues. Then

5.10a) $\qquad \rho(kP) = 1 - \frac{3}{2} k\pi^2 h^2 + O(h^3) \ ,$

5.10b) $\qquad \rho(k^2L) = 1 - \frac{3}{4} k\pi^2 h^2 + O(h^3) \ ,$

5.10c) $\qquad \rho(k^3B) = 1 - \frac{k}{2} \pi^2 h^2 + O(h^3) \ .$

<u>Proof</u>: The argument uses Theorem 4.1 and Theorem 3.1 and is like that given for the proof of Theorem 5.1.

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